

Noncanonical Quantization of Gravity. I. Foundations of Affine Quantum Gravity

John R. Klauder

Departments of Physics and Mathematics

University of Florida

Gainesville, Fl 32611

Abstract

The nature of the classical canonical phase-space variables for gravity suggests that the associated quantum field operators should obey affine commutation relations rather than canonical commutation relations. Prior to the introduction of constraints, a primary kinematical representation is derived in the form of a reproducing kernel and its associated reproducing kernel Hilbert space. Constraints are introduced following the projection operator method which involves no gauge fixing, no complicated moduli space, nor any auxiliary fields. The result, which is only qualitatively sketched in the present paper, involves another reproducing kernel with which inner products are defined for the physical Hilbert space and which is obtained through a reduction of the original reproducing kernel. Several of the steps involved in this general analysis are illustrated by means of analogous steps applied to one-dimensional quantum mechanical models. These toy models help in motivating and understanding the analysis in the case of gravity.

1 Introduction

General relativity is, in certain ways, fundamentally different than most other physically relevant classical field theories, and the same remark applies to attempts to provide associated quantum formulations. The space-time metric

$g_{\mu\nu}(x)$, $x \in \mathbb{R}^4$, $\mu, \nu = 0, 1, 2, 3$, possesses a signature requirement that is incompatible with the space of metrics being a linear vector space.¹ The inverse metric $g^{\sigma\mu}(x)$, defined so that $g^{\sigma\mu}(x)g_{\mu\nu}(x) = \delta^\sigma_\nu$, is classically trivial but it is quantum mechanically challenged since the left-hand side involves the product of two operator-valued distributions. Moreover, the spatial and temporal constraints that hold at each space-time point classically close algebraically, but they exhibit an anomaly when quantized. In effect, this fact changes the constraints from first class (classically) to second class (quantum mechanically). And, of course, there is the well-known fact that unlike other theories which take place on a fixed space-time stage, the theory of gravity involves the dynamics of the space-time stage itself. The purpose of this article is to discuss some basic issues surrounding quantum gravity from a viewpoint different than traditional ones.²

Let us outline the general approach we shall adopt. First, we focus on basic kinematics and the quantum theory of positive definite, 3×3 matrix-valued field variables and associated noncanonical “conjugate” field variables, designed to offer an initial class of coherent states and coherent-state induced Hilbert space representations. In this step it is noteworthy that, besides the signature issue for the metric, the existence of an operator $g^{jk}(x)$, $j, k = 1, 2, 3$, inverse to $g_{kl}(x)$ is shown, such that, when suitably defined, the equation $g^{jk}(x)g_{kl}(x) = \delta^j_k$ is fulfilled. Second, we introduce the spatial and temporal constraints in the projection operator approach recently developed by the author and others [4, 5, 6, 7]. This procedure has the advantage of working entirely with the classical degrees of freedom, including the c -number Lagrange multipliers—specifically, the lapse and shift functions. It is not necessary to introduce additional fields (e.g., ghosts with false statistics), nor choose gauges, nor pass to moduli spaces, etc. Initially, the constraints are imposed in a regularized fashion. Subsequently, the removal of the regularization is analyzed, a process which often involves an automatic change of Hilbert-space representation. Assuming that the limit removing the regularization exists, the physical Hilbert space that arises is then, generally speaking, best described as a reproducing kernel Hilbert space that emerges

¹Although, we assume a $3+1$ theory of gravity for illustrative purposes, it is straightforward to generalize to an $s+1$ theory as well, $s \geq 1$.

²The closest work in spirit to that discussed in this paper is that of the author [1], Isham and Kakas [2], and especially Pilati [3]. See Sec. 5 for an extensive discussion. We do not directly comment on current schemes for quantizing gravity.

from the reduction of the original reproducing kernel.

Before we undertake any discussion of gravity, however, we sketch in Sec. 2 the key concepts as applied to some simple, few degree-of-freedom systems. In Sec. 3 we construct a suitable kinematical framework for quantum gravity, while in Sec. 4 we analyze the introduction of constraints. Section 5 contains a general discussion about operator representations and constraints in relation to reparameterization invariance. In Part II of this work, the analysis of gravitational constraints is discussed in detail. In addition, the classical limit of the affine gravitational quantum theory developed in Secs. 3-4 will be discussed and compared with classical gravity.

2 Elementary Illustration of Key Concepts

As is generally well known, there is only one irreducible representation up to unitary equivalence of canonical, self-adjoint operators P and Q satisfying the Weyl form of the canonical commutation relations. This representation, equivalent to the Schrödinger representation, implies that the spectrum of both P and Q cover the whole real line. Such operator degrees of freedom are appropriate for many systems with a finite or an infinite number of degrees of freedom, but they are inappropriate for gravity. The reason for this is that the classical 3×3 metric is strictly positive definite and the associated quantum field operator cannot be represented by an operator whose spectrum is unbounded above and below. Instead, of the usual relation $[Q, P] = i$, with $\hbar = 1$, one is led to consider an *affine commutation relation* [8], which for a single degree of freedom takes the form

$$[Q, D] = iQ. \quad (1)$$

Here $D \equiv (PQ + QP)/2$ denotes the dilation operator, and it follows [9, 10] that solutions of the affine commutation relations exist with irreducible, self-adjoint operators D and Q for which—and this is the important part— $Q > 0$. (There are two other inequivalent self-adjoint solutions, one where $Q < 0$, which is rather like the representation of interest, and another for which $Q = 0$ [9]. Neither of these representations will be of interest in this article.) Even though the operator P is only a symmetric operator which has no self-adjoint extension, the introduction of the self-adjoint operator D provides the substitute commutation relation given above. These two commutation

relations are not in conflict since the affine commutation relation follows directly from the Heisenberg commutation relation simply by multiplication of the latter by Q . We note that an analog of the affine variables will be used in the case of the gravitational field to maintain the positivity of the local quantum field operator for the 3×3 spatial metric.

Continuing with the one-dimensional example, and based on self-adjoint operators that satisfy the affine commutation relation, let us introduce *affine coherent states* [11], $|p, q\rangle \in \mathfrak{H}$, defined by the expression

$$|p, q\rangle \equiv e^{ipQ} e^{-i \ln(q)D} |\eta\rangle, \quad -\infty < p < \infty, \quad 0 < q < \infty. \quad (2)$$

Here, the fiducial vector $|\eta\rangle$ is chosen to satisfy several conditions, which, using the shorthand $\langle(\cdot)\rangle \equiv \langle\eta|(\cdot)|\eta\rangle$, are specifically given by

$$\langle Q^{-1} \rangle \equiv C < \infty, \quad \langle \mathbb{1} \rangle = 1, \quad \langle Q \rangle = 1, \quad \langle D \rangle = 0, \quad \langle P \rangle = 0. \quad (3)$$

The first condition is required, while the remaining conditions are chosen for convenience. The coherent states also admit a resolution of unity [11] expressed in the form

$$\mathbb{1} = \int |p, q\rangle \langle p, q| d\tau(p, q), \quad d\tau(p, q) = dp dq / 2\pi C, \quad (4)$$

integrated over the half plane $\mathbb{R} \times \mathbb{R}^+$.

In particular, diagonalizing the self-adjoint operator $Q = \int_0^\infty x|x\rangle \langle x| dx$ in terms of standard Dirac-normalized eigenvectors, leads to a representation for the coherent-state overlap given by

$$\begin{aligned} \langle p, q|r, s \rangle &\equiv \langle \eta | e^{i \ln(q)D} e^{-ipQ} e^{irQ} e^{-i \ln(s)D} | \eta \rangle \\ &= (qs)^{-1/2} \int_0^\infty \eta(x/q)^* e^{-ix(p-r)} \eta(x/s) dx, \end{aligned} \quad (5)$$

where the fiducial function $\eta(x) = \langle x|\eta \rangle$ denotes the Schrödinger representation of the fiducial vector $|\eta\rangle$. It is important to observe, for some suitable function F , that

$$\langle p, q|r, s \rangle = F(q, p - r, s), \quad (6)$$

namely, that p and r universally enter in the form $p - r$. It is also clear that $\langle p, q|r, s \rangle$ defines a *continuous, positive-definite function*, which, apart from

the continuity, means that

$$\sum_{n,m=1}^N \alpha_n^* \alpha_m \langle p_n, q_n | p_m, q_m \rangle \geq 0 \quad (7)$$

for arbitrary complex $\{\alpha_n\}_{n=1}^N$ and real $\{p_n, q_n\}_{n=1}^N$ sequences, with $N < \infty$. The function (5) may be taken as the *reproducing kernel for a reproducing kernel Hilbert space* [12]. Note that the information in $\langle p, q | r, s \rangle$ is enough to recover $\eta(x)$ apart from an overall constant phase factor. Thus different fiducial functions (not related by a constant phase factor) generate distinct reproducing kernels. Since each reproducing kernel Hilbert space has one and only one reproducing kernel [12], it follows for different $\eta(x)$ that the Hilbert space functional realizations are completely disjoint except for the zero element. Basic elements of a dense set of vectors in each such Hilbert space are given by continuous functions of the form

$$\psi(p, q) \equiv \sum_{n=1}^N \alpha_n \langle p, q | p_n, q_n \rangle \quad (8)$$

defined for arbitrary complex $\{\alpha_n\}_{n=1}^N$, and real $\{p_n, q_n\}_{n=1}^N$ sequences, with $N < \infty$. Let a second such function be given by

$$\phi(p, q) \equiv \sum_{j=1}^J \beta_j \langle p, q | \bar{p}_j, \bar{q}_j \rangle \quad (9)$$

defined for arbitrary complex $\{\beta_j\}_{j=1}^J$ and real $\{\bar{p}_j, \bar{q}_j\}_{j=1}^J$ sequences, with $J < \infty$. The inner product of two such vectors is then *defined* [12] to be

$$\langle \psi | \phi \rangle \equiv (\psi(\cdot, \cdot) \text{ , } \phi(\cdot, \cdot)) \equiv \sum_{n=1}^N \sum_{j=1}^J \alpha_n^* \beta_j \langle p_n, q_n | \bar{p}_j, \bar{q}_j \rangle \text{ ,} \quad (10)$$

which when $|\phi\rangle = |\psi\rangle$ is, by definition, nonnegative. The resultant pre-Hilbert space is completed to a (reproducing kernel) Hilbert space \mathcal{C} by including all Cauchy sequences in the norm $\| |\psi\rangle \| \equiv \sqrt{\langle \psi | \psi \rangle}$ as N tends to infinity. Lastly, we note that the space of functions appropriate to one reproducing kernel is *identical* to the space of functions appropriate to a second

reproducing kernel that is just a constant multiple of the first reproducing kernel. This fact does not contradict the uniqueness of the reproducing kernel for each Hilbert space because strictly different inner products are assigned in the two cases. Of course, the foregoing discussion applies quite generally and is not limited to any one sort of reproducing kernel.

When the states $|p, q\rangle$ form a set of coherent states—as we assume in the present case—the inner product has an *alternative representation* given by a local integral of the form

$$\langle\psi|\phi\rangle = \int \psi(p, q)^* \phi(p, q) d\tau(p, q) , \quad (11)$$

expressed in terms of $\psi(p, q) \equiv \langle p, q|\psi\rangle$ and $\phi(p, q) \equiv \langle p, q|\phi\rangle$. It follows that this formula holds for all elements of the completed Hilbert space \mathcal{C} , and, moreover, every element of the so-completed space is a *bounded and continuous function*, the collection of which forms a rather special closed subspace of $L^2(\mathbb{R}^2, d\tau)$.

Thanks to the coherent-state resolution of unity, it follows that the coherent state overlap function satisfies the integral equation

$$\langle p'', q''|p', q'\rangle = \int \langle p'', q''|p, q\rangle \langle p, q|p', q'\rangle d\tau(p, q) , \quad (12)$$

a basic relation, which, if it was established as a first step for the continuous function $\langle p'', q''|p', q'\rangle [= \langle p', q'|p'', q''\rangle^*]$, guarantees the existence of a local integral representation for the inner product of two arbitrary elements in the associated reproducing kernel Hilbert space. Several useful properties follow from this reproducing property. For example, repeated use of the resolution of unity leads to the fact that

$$\langle p'', q''|p', q'\rangle = \lim_{L \rightarrow \infty} \int \cdots \int \prod_{l=0}^L \langle p_{l+1}, q_{l+1}|p_l, q_l\rangle \prod_{l=1}^L d\tau(p_l, q_l) , \quad (13)$$

in which we have identified $p'', q'' = p_{L+1}, q_{L+1}$ and $p', q' = p_0, q_0$. In turn, making an (unjustified!) interchange of the (continuum) limit with the integrations, and writing for the integrand the form it would assume for continuous and differentiable paths, gives rise to the suggestive but strictly formal expression [13]

$$\langle p'', q''|p', q'\rangle = \int e^{-i \int_0^T q(t) \dot{p}(t) dt} \mathcal{D}\tau(p, q) , \quad (14)$$

which determines a formal path integral representation for the kinematics that applies for any $T > 0$. Thus, the existence of a coherent state resolution of unity is the necessary condition to introduce a traditional coherent state phase-space path integral representation for the kinematics, which specifically leads to the reproducing kernel. A path integral for the kinematics is a necessary prerequisite to obtain a path integral for the dynamics. While less than ideal, the formal path integral itself may be used as a starting point for quantization. Even though the formal nature of the path integral renders it basically undefined, one may always (re)introduce a regularization by a lattice-limit formulation (as above), using suitable ingenuity to choose an acceptable integrand. This procedure is more or less standard by now.

However, it must be appreciated that reproducing kernels for reproducing kernel Hilbert spaces are *not required* to fulfill a (positive) local integral representation for the inner product. In cases where the integral for the resolution of unity does not exist, one must accept the inner product that is given directly by the reproducing kernel, which means that the integral relation (12) is replaced by

$$(\langle \cdot, \cdot | p'', q'' \rangle \text{ , } \langle \cdot, \cdot | p', q' \rangle) \equiv \langle p'', q'' | p', q' \rangle . \quad (15)$$

When this is the case we say that $\{|p, q\rangle\}$ forms a set of *weak coherent states* [14], i.e., the elements of $\{|p, q\rangle\}$ span the Hilbert space \mathfrak{H} , but do not admit a local integral representation for the inner product of elements in the associated reproducing kernel Hilbert space.

A simple example of a reproducing kernel Hilbert space without a local integral representation for the inner product is determined, for $u'', u' \in \mathbb{R}$, by the reproducing kernel $\langle u'' | u' \rangle \equiv \exp[-(u'' - u')^2]$; here, one must use $(\langle \cdot | u'' \rangle \text{ , } \langle \cdot | u' \rangle) \equiv \langle u'' | u' \rangle$, which is then extended by linearity and continuity to all Hilbert space vectors.

A more relevant set of examples is given by the following discussion applied to our simple model. Let $\alpha > -1/2$, and choose $\eta(x) \equiv N x^\alpha \exp(-\beta x)$. Here the factor N is fixed by requiring $\int_0^\infty |\eta(x)|^2 dx = 1$. The two conditions $\langle Q \rangle = 1$ and $\langle Q^{-1} \rangle = C < \infty$, lead to $\beta - \frac{1}{2} = \alpha > 0$; in this case $C = 1 - 1/(2\beta)$. In turn, the reproducing kernel is given explicitly [15] by

$$\langle p, q | r, s \rangle = \left[\frac{(qs)^{-1/2}}{\frac{1}{2}(q^{-1} + s^{-1}) + i\frac{1}{2}\beta^{-1}(p - r)} \right]^{2\beta}$$

$$= \exp(-2\beta \ln\{[\frac{1}{2}(q^{-1} + s^{-1}) + i\frac{1}{2}\beta^{-1}(p - r)]/(qs)^{-1/2}\}) ; \quad (16)$$

the second form is given for comparison purposes to the gravitational case. As long as $\beta > \frac{1}{2}$ it follows that the states $|p, q\rangle$ are a set of coherent states with a proper resolution of unity and therefore a local integral representation for the inner product exists. In this case, path integrals exist as lattice limits, and the whole situation seems familiar. On the other hand, if $0 < \beta \leq \frac{1}{2}$, the overlap function $\langle p, q|r, s\rangle$ defined above is still a positive-definite function and, therefore, it is a valid reproducing kernel which leads to an associated reproducing kernel Hilbert space; however, *such a Hilbert space does not admit a local integral representation for the inner product in terms of the given representatives*. Therefore, there is *no* conventional coherent state path integral for the kinematics, and thus also for the dynamics, in a reproducing kernel Hilbert space representation when $0 < \beta \leq \frac{1}{2}$. This lack of a conventional coherent state path integral representation may appear to be detrimental to any program to introduce quantization, dynamics, etc.—but there is hope.

There is another way to generate the reproducing kernel for the given family of fiducial vectors which may be applied for all $\beta > 0$. Let us first focus on $\beta > \frac{1}{2}$. In that case, observe, by construction and using $\partial_p \equiv \partial/\partial p$, etc., that for every $|\psi\rangle \in \mathfrak{H}$,

$$B\psi(p, q) \equiv \{-iq^{-1}\partial_p + 1 + \beta^{-1}q\partial_q\}\psi(p, q) = 0 , \quad (17)$$

an equation which represents a (complex) *polarization* [16] of $L^2(\mathbb{R}^2, d\tau)$. It follows that the second-order differential operator $A \equiv \frac{1}{2}\beta B^\dagger B \geq 0$, and therefore A can be used to generate a semigroup. In particular, for any $T > 0$ and as $\nu \rightarrow \infty$, the expression $e^{-\nu T A}$ becomes a *projection operator* onto the subspace \mathcal{C} of solutions to the polarization equation [17]. In a two degree of freedom Schrödinger representation—symbolized by $|p, q\rangle$, where $(p, q) \in \mathbb{R} \times \mathbb{R}^+$ and $(p, q|r, s) = \delta(p - r)\delta(q - s)$ —it follows, from a two-variable Feynman-Kac-Stratonovich path integral formula [18], that

$$\begin{aligned} \langle p'', q''|p', q'\rangle &\equiv \lim_{\nu \rightarrow \infty} (p'', q''|e^{-\nu T A}|p', q') \\ &= \lim_{\nu \rightarrow \infty} \mathcal{N} \int e^{-i\int_0^T q(t)\dot{p}(t)dt - (1/2\nu)\int_0^T [\beta^{-1}q(t)^2\dot{p}(t)^2 + \beta q(t)^{-2}\dot{q}(t)^2]dt} \mathcal{D}p \mathcal{D}q \end{aligned}$$

$$\equiv \lim_{\nu \rightarrow \infty} e^{\nu T/2} \int e^{-i \int_0^T q(t) dp(t)} dW^\nu(p, q) , \quad (18)$$

where W^ν denotes a two-dimensional Wiener measure with diffusion constant ν , pinned at $t = 0$ to p', q' and at time $t = T$ to p'', q'' , which is supported on a space of constant negative curvature $R = -2/\beta$. It is noteworthy in (18) that the variable $p(t)$ enters only in the form $\dot{p}(t)$, a fact which leads to the result depending only on the difference, $p'' - p'$. For every $\nu < \infty$, and with probability one, all Wiener paths in the given path integral are *continuous*, and, for purposes of coordinate transformations, it is convenient to adopt the (midpoint) Stratonovich rule to define the stochastic integral $-\int q(t) dp(t)$ since, in that case, the rules of the ordinary calculus hold. Such a representation is said to involve a *continuous-time regularization* [19].

Now we consider the case where $0 < \beta \leq \frac{1}{2}$. The solutions to (17) are, up to a factor, analytic functions, but they are no longer square integrable, as is clear from the fact that the large q behavior of (16) is controlled only by the factor $q^{-\beta}$. As a consequence, the operator $A \geq 0$ has only a continuous spectrum. The family of operators $e^{-\nu T A}$ is still a semigroup, and the expression $(p'', q'' | e^{-\nu T A} | p', q')$ still has the formal path integral representation given in the middle line of (18). However, as $\nu \rightarrow \infty$, $e^{-\nu T A}$ does not lead to a projection operator; instead we need to extract the germ of that semigroup as $\nu \rightarrow \infty$. If we let $E \geq 0$ denote continuum eigenvalues for the operator $\frac{1}{2} B^\dagger B$, we can write

$$(p'', q'' | e^{-\nu T A} | p', q') = \int (p'', q'' | E, v) e^{-\nu T \beta E} (E, v | p', q') \rho(E, v) dE dv , \quad (19)$$

for some density of states $\rho(E, v)$. Here, the variable v labels degeneracy for $\frac{1}{2} B^\dagger B$. For the sake of illustration, let us assume that $\rho(E, v) \simeq \overline{C} E^w \overline{\rho}(v)$, $w > -1$, for $E \ll 1$. We base this assumption on the fact that there is no reason for an E -dependent degeneracy for very tiny E . Thus, we consider the expression $J(\nu) \equiv (\nu \beta T)^{w+1} / \overline{C} \Gamma(w+1)$, and are led to the fact that

$$\begin{aligned} & \lim_{\nu \rightarrow \infty} J(\nu) (p'', q'' | e^{-\nu T A} | p', q') \\ &= \lim_{\nu \rightarrow \infty} \int (p'', q'' | E, v) e^{-\nu T \beta E} (E, v | p', q') J(\nu) \rho(E, v) dE \\ &= \int (p'', q'' | 0, v) (0, v | p', q') \overline{\rho}(v) dv . \end{aligned} \quad (20)$$

In effect, this procedure has enabled us to pass to the germ of the semigroup. Observe that the rescaling factor is independent of the coherent state labels, and thus we are only making a ν -dependent rescaling before the limit $\nu \rightarrow \infty$ is taken. If necessary, we can rescale our expression by letting $J(\nu) \rightarrow \overline{J}(\nu) = M(p'', q'') M(p', q') J(\nu)$ to achieve normalization without effecting its positive-definite character. In summary, for $0 < \beta \leq \frac{1}{2}$, we claim that instead of (18) we can write

$$\begin{aligned}
\langle p'', q'' | p', q' \rangle &\equiv \lim_{\nu \rightarrow \infty} \overline{J}(\nu) (p'', q'' | e^{-\nu T A} | p', q') \\
&= \lim_{\nu \rightarrow \infty} \overline{\mathcal{N}} \int e^{-i \int_0^T q(t) \dot{p}(t) dt - (1/2\nu) \int_0^T [\beta^{-1} q(t)^2 \dot{p}(t)^2 + \beta q(t)^{-2} \dot{q}(t)^2] dt} \mathcal{D}p \mathcal{D}q \\
&\equiv \lim_{\nu \rightarrow \infty} \overline{J}(\nu) e^{\nu T/2} \int e^{-i \int_0^T q(t) dp(t)} dW^\nu(p, q) ,
\end{aligned} \tag{21}$$

Convergence in this case is initially regarded in the sense of distributions. To lead to the desired result, we appeal to analyticity (up to a specific factor) of the result in $q^{-1} + i\beta^{-1}p$, analyticity (up to another factor) in the variable $s^{-1} - i\beta^{-1}r$, and dependence on $p - r$. Note that $\overline{J}(\nu)$ can always be determined self consistently by insisting that $\langle p, q | p, q \rangle = 1$ for all (p, q) . In simpler terms, we can always regard $\overline{J}(\nu)$ as part of the needed normalization coded into $\overline{\mathcal{N}}$ in the formal path integral expression.

As will become evident later, various features of reproducing kernels, reproducing kernel Hilbert spaces, and associated rules for defining inner products illustrated above will carry over into the quantum gravity case as well.

2.1 Operators and symbols

In addition to the properties of the reproducing kernel Hilbert space, certain *symbols* associated with operators are important. Let us introduce the upper symbol $H(p, q)$ associated to the operator $\mathcal{H}(P, Q)$ and defined, modulo suitable domain conditions, by the expression

$$H(p, q) \equiv \langle p, q | \mathcal{H}(P, Q) | p, q \rangle = \langle \mathcal{H}(p + P/q, qQ) \rangle . \tag{22}$$

For example, if $\mathcal{H}(P, Q) = P^2 - Q^{-1}$ denotes a quantum Hamiltonian, then $H(p, q) = p^2 + \langle P^2 \rangle / q^2 - C/q$. Since $C = O(1)$ (e.g., for large β) and $\langle Q \rangle = 1$,

it follows that $\langle P^2 \rangle = O(\hbar^2)$. Observe that H basically agrees with the expected classical Hamiltonian in the limit that $\hbar \rightarrow 0$, but prior to that limit H includes a quantum induced barrier to singularities in solutions of the usual classical equations of motion. We adopt the expression $H(p, q)$ as the (\hbar -augmented) classical Hamiltonian and refer to the connection between the quantum generator \mathcal{H} and the classical generator H as the *weak correspondence principle* [20]. In this way, the classical and quantum theories may both *coexist*, as they do in Nature.

There is also another set of symbols that are important. We introduce the lower symbol $h(p, q)$ which is related to the operator $\mathcal{H}(P, Q)$ by the relation

$$\mathcal{H}(P, Q) = \int h(p, q) |p, q\rangle \langle p, q| d\tau(p, q) . \quad (23)$$

For the one-parameter class of fiducial vectors leading to (16), it follows that a dense set of operators admit such a symbol for a reasonable set of functions.

In the quantum gravity case, there are twin goals: (i) to ensure that the field operators of interest are well defined and locally self adjoint in the given field operator representation; and (ii) to choose locally self adjoint constraint operators that have a weak correspondence principle which connects them with the desired form of the classical constraint generators (possibly \hbar augmented).

2.2 Imposition of constraints

We adopt the projection operator approach to the quantization of systems with constraints [4, 5, 6, 7]. Let $\{\Phi_\alpha(P, Q)\}_{\alpha=1}^A$, $A < \infty$, denote a set of constraints each given by a self-adjoint operator. Further assume that $\Phi \cdot \Phi \equiv \sum_{\alpha=1}^A (\Phi_\alpha)^2$ is also self adjoint. We define the (provisional) physical Hilbert space $\mathfrak{H}_{\text{phys}} \equiv \mathbb{E}\mathfrak{H}$, where $\mathbb{E} = \mathbb{E}^\dagger = \mathbb{E}^2$ is a uniquely defined projection operator, and in turn choose

$$\mathbb{E} \equiv \mathbb{E}(\Phi \cdot \Phi \leq \delta(\hbar)^2) , \quad (24)$$

where $\delta(\hbar)$ is *not* a δ -function but a small, positive, possibly \hbar dependent, *regularization parameter* for the set of constraints. As shown below, $\delta(\hbar)$ is chosen so that \mathbb{E} is the desired projection operator. This choice may entail a specific representation of the Hilbert space and a suitable limit as $\delta \rightarrow 0$ to extract the germ of the projection operator.

We may illustrate this latter situation for the constraint $\Phi \equiv Q - 1$, assuming initially that $\delta < 1$. In this case

$$\langle \psi | \mathbb{E}((Q - 1)^2 \leq \delta^2) | \phi \rangle = \int_{1-\delta}^{1+\delta} dx \int d\sigma(y) \psi(x, y)^* \phi(x, y) , \quad (25)$$

where σ accounts for any degeneracy that may be present. When restricted to functions ψ_o and ϕ_o in the dense set \mathfrak{D} , where

$$\mathfrak{D} \equiv \{ \text{polynomial}(x, y) e^{-x^2 - y^2} \} , \quad (26)$$

and rescaled by a suitable factor, the projection operator matrix elements lead to the expression

$$\begin{aligned} (2\delta)^{-1} \langle \psi_o | \mathbb{E}((Q - 1)^2 \leq \delta^2) | \phi_o \rangle \\ = (2\delta)^{-1} \int_{1-\delta}^{1+\delta} dx \int d\sigma(y) \psi_o(x, y)^* \phi_o(x, y) . \end{aligned} \quad (27)$$

Now, as $\delta \rightarrow 0$, this expression becomes

$$\int \psi_o(1, y)^* \phi_o(1, y) d\sigma(y) \equiv ((\psi_o, \phi_o)) . \quad (28)$$

Interpreting this final expression as a sequilinear form, one completes the desired Hilbert space by adding all Cauchy sequences in the associated norm $||| \psi_o ||| \equiv \sqrt{((\psi_o, \psi_o))}$. The result is the true physical Hilbert space in which the constraint $Q - 1 = 0$ is fulfilled, and this example illustrates how constraints are to be treated when $\Phi \cdot \Phi$ has its zero in the continuous spectrum.

A second example of an imposition of constraints is given by $\Phi_1 = Q - 1$, as before, along with $\Phi_2 = D$. This situation corresponds to second class constraints, and serves as a simple qualitative model of what occurs in the gravitational case. In this case

$$\mathbb{E} = \mathbb{E}(D^2 + (Q - 1)^2 \leq \delta(\hbar)^2) . \quad (29)$$

Here, the left-hand side of the argument can be regarded as a ‘‘Hamiltonian’’, and the ground state $|0'\rangle$ for such a system (nondegenerate, in the present example) can be sought. In particular, there are two positive parameters, δ' and δ'' , such that, for all δ with $\delta' \leq \delta < \delta''$, then

$$\mathbb{E} = \mathbb{E}(D^2 + (Q - 1)^2 \leq \delta(\hbar)^2) \equiv |0'\rangle \langle 0'| . \quad (30)$$

This is the desired choice to make for \mathbb{E} in the case where $\Phi \cdot \Phi$ has a discrete spectrum near zero that does not include zero.

For completeness, if $\Phi \cdot \Phi$ has a discrete spectrum including zero, then it suffices to choose

$$\mathbb{E} = \mathbb{E}(\Phi \cdot \Phi = 0) = \mathbb{E}(\Phi \cdot \Phi \leq \delta(\hbar)^2) , \quad (31)$$

where in the present case $\delta(\hbar) > 0$ is chosen small enough to include only the subspace for which $\Phi \cdot \Phi = 0$. This simple case does not seem to arise in quantum gravity. See [4, 7] for a discussion of gauge invariance.

When the number of constraints is infinite, $A = \infty$, as will be the case for a field theory, then a slightly different approach is appropriate. One form this takes is dealt with in Sec. 4.

2.3 Appearance of time

In a reparametrization invariant problem in quantum mechanics it is typical that dynamics is cast in the guise of kinematics at the expense of introducing an additional degree of freedom plus a first-class constraint; see, e.g., [21]. Let the resultant kinematical reproducing kernel with the extra degree of freedom be given by $\langle p'', q'', s'', t'' | \mathbb{E} | p', q', s', t' \rangle$, where \mathbb{E} is the projection operator enforcing the constraint. Next, reduce this expression, for example, as in the procedure

$$\langle p'', q'', t'' | p', q', t' \rangle \equiv \int \int \langle p'', q'', s'', t'' | \mathbb{E} | p', q', s', t' \rangle ds'' ds' . \quad (32)$$

The result is a new positive-definite function that can be used to define a reproducing kernel Hilbert space. However, it may well happen that the following identity holds

$$\begin{aligned} \langle p'', q'', t'' | p', q', t' \rangle &= (\langle \cdot, \cdot, \cdot | p'', q'', t'' \rangle \text{ } _J \langle \cdot, \cdot, \cdot | p', q', t' \rangle) \\ &= (\langle \cdot, \cdot, t | p'', q'', t'' \rangle \text{ } _J \langle \cdot, \cdot, t | p', q', t' \rangle) . \end{aligned} \quad (33)$$

This equation means that the space spanned by the states $|p, q, t\rangle$ by varying p, q and t is the *same space* spanned, by the same states, by varying p and q but with t held *fixed* at some value (e.g., $t = 0$). This situation implies that the states $|p, q, t\rangle$ are *extended coherent states* in the sense of [22], and, in particular, thanks to using canonical group coordinates in the

coherent-state parameterization, that $|p, q, t\rangle = \exp(-i\mathcal{H}t)|p, q, 0\rangle$ for some self-adjoint “Hamiltonian” \mathcal{H} . The parameter t is then recognized as the “time” t . For an explicit example of how this procedure works in detail see [23].

It is expected that a suitable time parameter will emerge in the extension of these ideas to the gravitational case.

2.4 Matrix generalization

Our preceding analysis has been confined to a single p and q and the associated affine quantum operators. In any generalization to the gravitational case, it will first be necessary to generalize the preceding discussion to 3×3 (or more generally to $s \times s$) matrix degrees of freedom and repeat an analysis similar to that of the present section. We do not include this discussion here since that is the subject of a separate work [24]. It is safe to say, apart from some technical details, that there are no special surprises in this generalization, and the basic concepts that we shall need are already present in the simplest case on which we have concentrated.

3 Gravitational Kinematics

3.1 Preliminaries

Let us start with the introduction of a three-dimensional *topological space* \mathcal{S} which locally is isomorphic to a subset of \mathbb{R}^3 . Locally, we generally use three “spatial” coordinates, say x^j , $j = 1, 2, 3$, to label a point in \mathcal{S} . This labeling is nonsingular and thus one-to-one. Whether a single coordinate chart covers \mathcal{S} depends on the global topological structure of \mathcal{S} . Let us fix this global topological structure from the outset—for example, topologically equivalent to \mathbb{R}^3 , S^3 , T^3 , etc. The theory of quantum gravity developed here does not engender topological changes of the underlying topological space \mathcal{S} . Note this lack of topological change applies only to the space \mathcal{S} . It is unrelated to any presumed “space” and/or “topology” associated with any quantum metric tensor, which, after all, is typically distributional in character.

If the space \mathcal{S} is such that more than one coordinate patch is required we arrange for the necessary matching conditions and rename the coordinates

within each patch by x^j , $j = 1, 2, 3$, for some domain. We can also consider alternative coordinates, say \bar{x}^j , $j = 1, 2, 3$, which are also nonsingular. We admit only differentiable coordinate transformations such that the Jacobian $[\partial x / \partial \bar{x}] \equiv \det(\partial x^j / \partial \bar{x}^k) \neq 0$ everywhere. The group composed of such invertible coordinate transformations is the *diffeomorphism group*.

We can also introduce functions on the space \mathcal{S} which in coordinate form may be denoted by $f(x)$. A scalar function is one for which $\bar{f}(\bar{x}) = f(x)$, while a scalar density of weight one satisfies $\bar{b}(\bar{x}) = [\partial x / \partial \bar{x}] b(x)$, or stated as a volume form, $dV = \bar{b}(\bar{x}) d^3\bar{x} = b(x) d^3x$. Observe that it is not necessary to have a metric in order to have a volume form. Whether \mathcal{S} is compact or noncompact, we assume that $0 < b(x) < \infty$ for all x and therefore $0 < b(x)^{-1} < \infty$ for all x as well. These properties are still valid after a nonsingular coordinate change. Integrals of a scalar function take the form $\int f dV = \int f(x) b(x) d^3x$ and are *invariant* under any coordinate transformation in the diffeomorphism group.

In the ADM (Arnowitt, Deser, Misner) [25] canonical formulation of classical gravity there are two fundamental fields $g_{kl}(x)$ [= $g_{lk}(x)$] and $\pi^{kl}(x)$ [= $\pi^{lk}(x)$]. The metric $g_{kl}(x)$ transforms as a (two-valent covariant) tensor, while the momentum $\pi^{kl}(x)$ transforms as a (two-valent contravariant) tensor density of weight one. Thus $\int g_{kl}(x) \pi^{kl}(x) d^3x$ [or even $\int g_{kl}(x, t) \dot{\pi}^{kl}(x, t) d^3x dt$] is an invariant under diffeomorphism group transformations (on the spatial hyperspace, of course). Note well: The latter example pertains to a generalization of the former one including an additional independent variable t which possibly could be identified with (coordinate) “time”.

A metric is not an arbitrary tensor but is restricted to be positive definite. Specifically, for any real α^j , $j = 1, 2, 3$, where $\Sigma_{j=1}^3 (\alpha^j)^2 > 0$, it follows that $\alpha^k g_{kl}(x) \alpha^l > 0$ for all x . As a consequence, the positive-definite (two-valent contravariant) tensor $g^{kl}(x)$ exists at each point and is defined so that $g^{kl}(x) g_{lm}(x) = \delta_m^k$. In addition, $\sqrt{g(x)} \equiv \sqrt{\det[g_{kl}(x)]} > 0$ transforms as a scalar density of weight one. Thus, as is well known, $\sqrt{g(x)} d^3x$ characterizes a volume form, but this choice ties the volume form to a specific metric, or at least to a specific class of metrics. This close association to specific metrics is something we would like to avoid, and it leads us to choose $b(x) d^3x$ as the preferred volume form. Of course, if $b(x) = \sqrt{g(x)}$ everywhere in any coordinate system, then the volume form $b(x) d^3x$ is identical to the one based on a metric space and given by $\sqrt{g(x)} d^3x$.

3.2 Reproducing kernel—original Hilbert space

A study of canonical quantum gravity begins with the introduction of metric and momentum local quantum field operators, which we denote by $\sigma_{kl}(x)$ [= $\sigma_{lk}(x)$] and $\mu^{kl}(x)$ [= $\mu^{lk}(x)$], respectively. For such fields one postulates the canonical commutation relations

$$\begin{aligned} [\sigma_{kl}(x), \mu^{rs}(y)] &= i \delta_{kl}^{rs} \delta(x, y) , \\ [\sigma_{kl}(x), \sigma_{rs}(y)] &= 0 , \\ [\mu^{kl}(x), \mu^{rs}(y)] &= 0 , \end{aligned} \tag{34}$$

with $\delta_{kl}^{rs} \equiv (\delta_k^r \delta_l^s + \delta_l^r \delta_k^s)/2$. Since the right-hand side of the first equation is a tensor density of weight one, it is consistent that we define $\sigma_{kl}(x)$ to be a tensor and $\mu^{rs}(x)$ to be a tensor density of weight one. However, just as its one-dimensional counterpart, there are no local self-adjoint field and momentum operators that satisfy the canonical commutation relations as well as the requirement that $\{\sigma_{kl}(x)\} > 0$. To arrive at a suitable substitute set of commutation relations, we introduce, along with the local metric field operator $\sigma_{kl}(x)$, the local “scale” field operator $\kappa_k^r(x)$ which together obey the *affine commutation relations* [1, 2, 3]

$$\begin{aligned} [\kappa_k^r(x), \kappa_l^s(y)] &= i \frac{1}{2} [\delta_k^s \kappa_l^r(x) - \delta_l^r \kappa_k^s(x)] \delta(x, y) , \\ [\sigma_{kl}(x), \kappa_s^r(y)] &= i \frac{1}{2} [\delta_k^r \sigma_{ls}(x) + \delta_l^r \sigma_{ks}(x)] \delta(x, y) , \\ [\sigma_{kl}(x), \sigma_{rs}(y)] &= 0 . \end{aligned} \tag{35}$$

In these relations, $\sigma_{kl}(x)$ remains a tensor, while $\kappa_s^r(x)$ is a tensor density of weight one under coordinate transformations. The local operators $\kappa_s^r(x)$ are generators of the $\text{GL}(3, \mathbb{R})^\infty$ group [2, 3], while the local operators $\sigma_{kl}(x)$ are commuting “translations” coupled with the $\text{GL}(3, \mathbb{R})^\infty$ group by a semi-direct product. The given affine commutation relations are the natural generalization of the one-dimensional affine commutation relation presented in (1). In the case of one degree of freedom, the affine commutation relations follow from the canonical ones, while in the case of fields this is, strictly speaking, incorrect. It is true that

$$\kappa_k^r(x) = \frac{1}{2} [\sigma_{kl}(x) \mu^{lr}(x) + \mu^{rl}(x) \sigma_{lk}(x)]_R , \tag{36}$$

where the subscript R denotes an infinite multiplicative renormalized product to be defined later. However, the presence of an infinite rescaling means

that either the canonical or the affine set of commutation relations can hold, but not both at the same time. Since it is the affine commutation relations that are consistent with local self-adjoint operator solutions enjoying metric positivity, we shall adopt the noncanonical affine commutation relations. The choice of the affine commutation relations means that the canonical commutation relations do *not* hold, and therefore we are dealing with a *noncanonical* quantization of the gravitational field.

Accepting the affine field operators as generators, we introduce a primary set of normalized affine coherent states each of which—in a deliberate abuse of notation—is defined by

$$|\pi, g\rangle \equiv e^{i\int \pi^{kl}(x) \sigma_{kl}(x) d^3x} e^{-i\int \gamma_s^r(x) \kappa_r^s(x) d^3x} |\eta\rangle, \quad (37)$$

for a suitable fiducial vector $|\eta\rangle$ characterized below. In $|\pi, g\rangle$ the argument “ π ” denotes the momentum matrix field π^{ab} while “ g ” denotes the metric matrix field g_{ab} . By all rights, the states in question should have been called $|\pi, \gamma\rangle$, but as we shall see, the overlap of two such states, *for the featured choice of $|\eta\rangle$* [cf. (70)], depends only on the matrix (for each point x) $g \equiv \exp(\gamma^T/2) \exp(\gamma/2) \equiv \{g_{ab}\}$, where T means “transpose”.³ If the space \mathcal{S} is noncompact, then, as smooth c -number fields, both π and γ should go to zero sufficiently fast so that the indicated smeared field operators are indeed self-adjoint operators and generate unitary transformations as required. On the other hand, as we shall shortly see, this asymptotic behavior can, effectively, be significantly relaxed.

The overlap of two such coherent states leads to an expression of the form

$$\langle \pi'', g'' | \pi', g' \rangle = F(g'', \pi'' - \pi', g') \quad (38)$$

for some continuous functional F which depends only on the difference of the fields, $\pi''(x) - \pi'(x)$, an analog of which already occurred for the one-dimensional example. Whatever choice is made for the fiducial vector $|\eta\rangle$, the coherent state overlap function defines a continuous, positive-definite functional which, therefore, defines a reproducing kernel and its associated (separable) reproducing kernel Hilbert space \mathcal{C} . By construction, therefore,

³Observe by this parametrization that $g > 0$ as opposed to a traditional triad for which $g \geq 0$. To see that this distinction may possibly make a real difference see [15]. We remark that the nonsymmetric matrix $\exp(\gamma/2)$ would have relevance for spinor fields.

the set of coherent states $\{|\pi, g\rangle\}$ span the Hilbert space \mathfrak{H} . As such they form a basis (overcomplete to be sure!) for \mathfrak{H} . Based on arguments to follow, we are led to the proposal [cf., (16)] that

$$\begin{aligned} & \langle \pi'', g'' | \pi', g' \rangle = \\ & \exp \left(-2 \int b(x) d^3x \ln \left\{ \frac{\det \left\{ \frac{1}{2} [g''^{kl}(x) + g'^{kl}(x)] + i \frac{1}{2} b(x)^{-1} [\pi''^{kl}(x) - \pi'^{kl}(x)] \right\}}{(\det[g''^{kl}(x)])^{1/2} (\det[g'^{kl}(x)])^{1/2}} \right\} \right) \end{aligned} \quad (39)$$

This equation is central to our analysis of quantum gravity.

The coherent-state overlap (39) may be read in two qualitatively different ways. Although arrived at on the basis that $\pi''(x), \pi'(x) \rightarrow 0$ and $\gamma''(x), \gamma'(x) \rightarrow 0$ as $|x| \rightarrow \infty$, the given expression exists for a far wider limiting behavior. In particular, suppose there is a *fixed asymptotic behavior* such that for both $\pi = \pi''$ and $\pi = \pi'$, $\pi^{ab}(x) - \tilde{\pi}^{ab}(x) \rightarrow 0$ and for both $g = g''$ and $g = g'$, $g_{ab}(x) - \tilde{g}_{ab}(x) \rightarrow 0$, all terms vanishing sufficiently fast as $|x| \rightarrow \infty$. In this case $g \equiv \exp(\gamma^T/2) \tilde{g} \exp(\gamma/2)$. Note that the asymptotic fields can depend on x . In this case the coherent-state overlap still holds in the form given. This kind of asymptotic behavior reflects a change of the fiducial vector $|\eta\rangle$, which now depends on the explicitly chosen asymptotic form for the momentum and metric—or, equivalently, as we effectively do, one can hold $|\eta\rangle$ fixed and change the representations of the operator [cf., (71)]. By choosing a suitable asymptotic momentum and metric one can, in effect, redefine the topology of the underlying space \mathcal{S} . However, for simplicity, we shall assume simple Euclidean-like asymptotic behavior of the momentum and metric [$\tilde{\pi}^{ab}(x) \equiv 0$ and $\tilde{\gamma}_s^r(x) \equiv 0$, i.e., $\tilde{g}_{ab}(x) \equiv \delta_{ab}$].

A second way to study the coherent-state overlap is under coordinate transformations. Observe that $\langle \pi'', g'' | \pi', g' \rangle$ is *invariant* if, everywhere, we make the replacements

- (i) $b(x) d^3x$ by $\bar{b}(\bar{x}) d^3\bar{x} = b(x) d^3x$,
- (ii) $g^{kl}(x)$ by $\bar{g}^{kl}(\bar{x}) = M_r^k(x) g^{rs}(x) M_s^l(x)$,
- (iii) $b(x)^{-1} \pi^{kl}(x)$ by $\bar{b}(\bar{x})^{-1} \bar{\pi}^{kl}(\bar{x}) = b(x)^{-1} M_r^k(x) \pi^{rs}(x) M_s^l(x)$,

all for an arbitrary nonsingular matrix $M \equiv \{M_r^k\}$, $M_r^k(x) \equiv (\partial \bar{x}^k / \partial x^r)(x)$, that arises from a nonsingular coordinate transformation $x \rightarrow \bar{x} = \bar{x}(x)$. It suffices to restrict attention to those coordinate transformations continuously connected to the identity. When \mathcal{S} is compact, a wide class of M is allowed;

when \mathcal{S} is noncompact, the allowed elements M must also map coherent states into coherent states for the same fiducial vector. This restriction excludes any connection by coordinate transformations of two field sets with fundamentally different asymptotic behavior; such fields live in disjoint sets. The invariance under admissible coordinate transformations is symbolized by the statement that

$$\langle \bar{\pi}'', \bar{g}'' | \bar{\pi}', \bar{g}' \rangle = \langle \pi'', g'' | \pi', g' \rangle \quad (40)$$

for all suitable M . Since the allowed M form a representation of the connected component of the diffeomorphism group, it follows from this identity, and suitable continuity, that for sufficiently restricted M , the transformation $|\pi, g\rangle \rightarrow |\bar{\pi}, \bar{g}\rangle$ is induced by a *unitary transformation*, specifically that

$$|\bar{\pi}, \bar{g}\rangle \equiv U(M) |\pi, g\rangle, \quad (41)$$

$$U(M) \equiv \exp[-i \int N^j(x) \mathcal{H}_j(x) d^3x]. \quad (42)$$

Here $\mathcal{H}_j(x)$ denotes a local operator tensor density of weight one while $N^j(x)$ denotes a c -number tensor with sufficiently rapid decay at spatial infinity. Furthermore, using the shorthand that $\int N^j H_j \equiv \int N^j(y) H_j(y) d^3y$, the connection between M and N^j is implicitly given by

$$\begin{aligned} M_r^a g^{rs} M_s^b &= g^{ab} - \left\{ \int N^j H_j, g^{ab} \right\} + \frac{1}{2!} \left\{ \int N^k H_k, \left\{ \int N^j H_j, g^{ab} \right\} \right\} + \dots \\ &\equiv e^{-\left\{ \int N^j H_j, \cdot \right\}} g^{ab}, \end{aligned} \quad (43)$$

where $\{\cdot, \cdot\}$ denotes the classical Poisson brackets, and specifically, e.g., $\{g_{ab}(x), \pi^{rs}(y)\} = \delta_{ab}^{rs} \delta(x, y)$. In this expression, $H_j(x) = -g_{jk}(x) \pi^{kl}|_l(x)$, $j = 1, 2, 3$, where $(\cdot)|_l$ is the covariant derivative with respect to the 3×3 metric, denotes the classical generators of the diffeomorphism group [26]. The relationship of $H_j(y)$ and $\mathcal{H}_j(y)$ may be determined as follows. Expansion of the relation

$$\langle \pi'', g'' | e^{-i \int N^j(x) \mathcal{H}_j(x) d^3x} | \pi', g' \rangle = \langle \pi'', g'' | \bar{\pi}', \bar{g}' \rangle \quad (44)$$

to first order in N^j leads to

$$\langle \pi'', g'' | \int N^j(x) \mathcal{H}_j(x) d^3x | \pi', g' \rangle / \langle \pi'', g'' | \pi', g' \rangle$$

$$\begin{aligned}
&= -i \int b(x) d^3x \left([g''^{kl}(x) + g'^{kl}(x)] + i b(x)^{-1} [\pi''^{kl}(x) - \pi'^{kl}(x)] \right)^{-1} \\
&\quad \times [\delta g'^{kl}(x) - i b(x)^{-1} \delta \pi'^{kl}(x)] \\
&\quad + i \frac{1}{2} \int b(x) d^3x g'_{kl}(x) \delta g'^{kl}(x) ,
\end{aligned} \tag{45}$$

where

$$\delta g'^{kl}(x) \equiv g'^{kl}_{,j}(x) N^j(x) - g'^{jl}(x) N^k_{,j}(x) - g'^{kj}(x) N^l_{,j}(x) , \tag{46}$$

and likewise for $\delta \pi'^{kl}(x)$. This relation determines the coherent state matrix elements of $\mathcal{H}_j(x)$. Finally, observe that the diagonal coherent state matrix elements read

$$\langle \pi, g | \mathcal{H}_j(x) | \pi, g \rangle = -g_{jk}(x) \pi^{kl}_{|l}(x) \tag{47}$$

in conformity with the weak correspondence principle.

3.3 Path integral construction

If the given coherent states $|\pi, g\rangle$ possessed a resolution of unity, namely a nonnegative measure $\rho(\pi, g)$ (countably or even finitely additive) such that

$$\int \langle \pi'', g'' | \pi, g \rangle \langle \pi, g | \pi', g' \rangle d\rho(\pi, g) = \langle \pi'', g'' | \pi', g' \rangle , \tag{48}$$

then the construction of a path integral for the reproducing kernel would be straightforward and would follow the pattern illustrated in Sec. 2 for a single degree of freedom. However, for the proposed reproducing kernel $\langle \pi'', g'' | \pi', g' \rangle$ given in (39) no such measure exists and thus the traditional resolution of unity is unavailable. Consequently, as defined, $\{|\pi, g\rangle\}$ is a set of weak coherent states.

A similar kind of problem arose in the simple model discussed in Sec. 2 (when $0 < \beta \leq \frac{1}{2}$). In that case, the construction of a path integral representation proceeded in an alternative manner beginning first with a polarization. We assert that each of the given Hilbert space representatives,

$$\psi(\pi, g) \equiv \langle \pi, g | \psi \rangle = \sum_{n=1}^N \alpha_n \langle \pi, g | \pi_n, g_n \rangle \in \mathcal{C} , \tag{49}$$

satisfies the functional differential equation [cf., (17)]

$$B_s^r(x) \psi(\pi, g) \equiv \left[-ig^{rt}(x) \frac{\delta}{\delta \pi^{ts}(x)} + \delta_s^r + b(x)^{-1} g_{st}(x) \frac{\delta}{\delta g_{tr}(x)} \right] \psi(\pi, g) = 0 \quad (50)$$

for all spatial points x . Next, let us introduce the operator

$$\mathcal{A} \equiv \frac{1}{2} \int B_s^r(x)^\dagger B_s^r(x) b(x) d^3x, \quad (51)$$

and observe that $\mathcal{A} \geq 0$. Thus, with $T > 0$ and as $\nu \rightarrow \infty$, it follows that $\overline{\mathcal{J}}(\nu)e^{-\nu T \mathcal{A}}$, for some $\overline{\mathcal{J}}(\nu)$, serves to select out the subspace where (50) is fulfilled. Just as in the toy model of Sec. 2, the operator \mathcal{A} is a second order (functional) differential operator, and, as a consequence, a Feynman-Kac-Stratonovich path (i.e., functional) integral representation may be introduced. In particular, we obtain the formal expression [cf., (21)]

$$\begin{aligned} \langle \pi'', g'' | \pi', g' \rangle &= \lim_{\nu \rightarrow \infty} \overline{\mathcal{N}} \int \exp[-i \int g_{ab} \dot{\pi}^{ab} d^3x dt] \\ &\times \exp\{-(1/2\nu) \int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] d^3x dt\} \\ &\times \prod_{x,t} \prod_{a \leq b} d\pi^{ab}(x, t) dg_{ab}(x, t). \end{aligned} \quad (52)$$

Here, let us interpret t , $0 \leq t \leq T$, as coordinate “time”. On the right-hand side the canonical fields are functions of space and time, that is

$$g_{ab} = g_{ab}(x, t), \quad \pi^{ab} = \pi^{ab}(x, t); \quad (53)$$

the overdot ($\dot{}$) denotes a partial derivative with respect to t , and the integration is subject to the boundary conditions that $\pi(x, 0)$, $g(x, 0) = \pi'(x)$, $g'(x)$ and $\pi(x, T)$, $g(x, T) = \pi''(x)$, $g''(x)$. Observe that the field π enters this path integral expression only in the form $\dot{\pi}$; this fact is responsible for the result of the path integral depending only on $\pi'' - \pi'$. It is important to note, for any $\nu < \infty$, that underlying the formal measure given above, there is a genuine, countably additive measure on (generalized) functions g_{kl} and π^{rs} . Loosely speaking, such functions have Wiener-like behavior with respect to time and δ -correlated, generalized Poisson-like behavior with respect to space.

While (52) is invariant under spatial diffeomorphisms, it is less evident that it is also invariant under transformations of the time coordinate (by

itself).⁴ Formally speaking, the role of the limit $\nu \rightarrow \infty$ is to remove the effects of the continuous-time regularization. It is clear, however, that there is no need that removing those effects must be done in a *uniform* way independent of x . Thus we may replace ν by $\nu N(x)$ —now under the integral sign—where $N(x)$, $0 < N(x) < \infty$, is smooth and reflects the relative rate at which the regularization is removed at different spatial points. The end result is invariant under such a change. Moreover, at each point x we can run the process with different “clock” rates, i.e., $N(x) dt \rightarrow N(x, t) dt$ so long as the elapsed time is qualitatively unchanged. This remark means that we can choose any smooth *lapse function* $N(x, t)$, $0 < N(x, t) < \infty$, with the consequence that

$$\begin{aligned} \langle \pi'', g'' | \pi', g' \rangle &= \lim_{\nu \rightarrow \infty} \mathcal{N}' \int \exp[-i \int g_{ab} \dot{\pi}^{ab} d^3x dt] \\ &\times \exp\{-(1/2\nu) \int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] N(x, t)^{-1} d^3x dt\} \\ &\times \prod_{x, t} \prod_{a \leq b} d\pi^{ab}(x, t) dg_{ab}(x, t) . \end{aligned} \quad (54)$$

The necessary conditions for this more general expression to hold are, for all T , $0 < T < \infty$, and at all x , that

$$\int_0^T N(x, t) dt < \infty , \quad (55)$$

$$\int_0^\infty N(x, t) dt = \infty . \quad (56)$$

In this sense we observe that our formal path integral representation (52) for the coherent-state overlap is actually *invariant* under transformations of the time coordinate.

3.4 Metrical quantization

The formal, ν -dependent, weighting factor in the path integral expression (52) involves a *metric* $d\Sigma^2$ on the classical phase space which may be read out of the expression

$$d\Sigma^2/dt^2 = \int [b^{-1} g_{kl} g_{rs} \dot{\pi}^{lr} \dot{\pi}^{sk} + b g^{kl} g^{rs} \dot{g}_{lr} \dot{g}_{sk}] d^3x . \quad (57)$$

⁴The author thanks A. Ashtekar for raising the question of temporal transformation properties.

As presented, this expression for $d\Sigma^2$ is a *derived* quantity. Alternatively, it is clear that one could *start* the analysis by *postulating* a specific functional form for $d\Sigma^2$ to be used in a continuous-time regularization in the path integral construction of the reproducing kernel, and finally, by appealing to the GNS (Gel'fand, Naimark, Segal) Theorem [27], to recover the representation of the local field operators $\sigma_{kl}(x)$ and $\kappa_s^r(x)$. Adopting a metric on the classical phase space as the first step in a quantization procedure is called *metrical quantization* [28]. To carry out such a scheme for gravity, it is necessary that any postulated $d\Sigma^2$ satisfy several properties. First, it must be diffeomorphism *invariant* and second, on physical grounds, it should only depend on $d\pi^{kl}$ (or $\dot{\pi}^{kl}$ for $d\Sigma^2/dt^2$) and not on π^{kl} itself. Hence, we are initially led to consider

$$d\Sigma^2/dt^2 = \int [b^{-2} L_{abcd} \dot{\pi}^{bc} \dot{\pi}^{da} + M^{abcd} \dot{g}_{bc} \dot{g}_{da}] b(x) d^3x \quad (58)$$

for suitable, positive-definite tensors L and M constructed just from g_{kl} . The given choice for L and M , i.e., $L_{abcd} = \frac{1}{2}[g_{ab}g_{cd} + g_{ac}g_{bd}]$ and $M^{abcd} = \frac{1}{2}[g^{ab}g^{cd} + g^{ac}g^{bd}]$ satisfy $M = L^{-1}$ as matrices. This choice is very natural and moreover is identical to the form suggested by the study of certain $GL(3, \mathbb{R})$ coherent states for a 3×3 positive-definite matrix degree of freedom [24].

Nevertheless, in a metric-first quantization scheme, it is appropriate to examine other choices as well. For example, a term such as $b(x)^{-1} \dot{g}_{ab}(x) \dot{\pi}^{ab}(x)$ might be included, but this term may be eliminated by a translation of the momentum. Additionally, one may consider nonlocal contributions involving, for example, the term $\dot{g}_{bc}(x) \dot{g}_{da}(y)$, together with a kernel $K(x, y)$ specifying the interrelationship of the field at y to the field at x . However, no satisfactory solution for $K(x, y)$ other than one proportional to $\delta(x, y)$ will lead to an expression for $d\Sigma^2$ that is *invariant* under all diffeomorphisms. In point of fact, the possible choices for L and M are rather limited, especially when one requires that the (formal) integration measure at each point is canonical and thus has the form $\Pi_{a \leq b} d\pi^{ab} dg_{ab}$. For example, for $\lambda > 0$, let us consider the proposal that

$$L_{abcd}(\lambda) \equiv \frac{1}{2}[g_{ab}g_{cd} + g_{ac}g_{bd} + (\lambda - 1)g_{bc}g_{da}] . \quad (59)$$

Then in order to lead to a canonical integration measure it would be necessary that

$$M^{abcd}(\lambda) \equiv \frac{1}{2}[g^{ab}g^{cd} + g^{ac}g^{bd} + (\lambda^{-1} - 1)g^{bc}g^{da}] . \quad (60)$$

Only for $\lambda = 1$ is $M = L^{-1}$ which is just the choice we have made. (The form for the DeWitt metric [29], where $\lambda = -1$, is excluded because we require that L and M be positive definite.)

The preceding discussion has rather convincingly suggested the specifically chosen functional form for $d\Sigma^2$ —apart from one issue. It may seem even more natural to choose [30]

$$d\Sigma^2/dt^2 = \int [g^{-1/2} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + g^{1/2} g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] d^3x \quad (61)$$

rather than the choice we have made. This is a natural choice from a classical point of view, but it is less satisfactory from a quantum point of view. In either case, observe that a path integral such as (52) involves fields with $3+1$ independent variables; however, there are no *space derivatives* involved, only *time derivatives*. Such a model is known as an *ultralocal quantum field theory*, and by now there is much that is known about the rigorous construction and evaluation of such nontrivial (i.e., non-Gaussian) functional integrals through the study, for example, of ultralocal scalar quantum fields [31]. It is through the analysis of the gravitational models as ultralocal quantum field theories that the metric (61) is ruled out; for a simple reason, see Section 5.

3.5 Operator realization

In order to realize the metric and scale fields as quantum operators in a Hilbert space, \mathfrak{H} , it is expedient to introduce a set of conventional local *annihilation and creation operators*, $A(x, k)$ and $A(x, k)^\dagger$, respectively, with the only nonvanishing commutator given by

$$[A(x, k), A(x', k')^\dagger] = \delta(x, x') \delta(k, k') \mathbb{1} , \quad (62)$$

where $\mathbb{1}$ denotes the unit operator. Here, $x \in \mathbb{R}^3$, while $k \equiv \{k_{rs}\}$ denotes a positive-definite, 3×3 matrix degree of freedom confined to the domain where $\{k_{rs}\} > 0$. We introduce a “no-particle” state $|0\rangle$ such that $A(x, k) |0\rangle = 0$ for all arguments. Additional states are determined by suitably smeared linear combinations of

$$A(x_1, k_1)^\dagger A(x_2, k_2)^\dagger \cdots A(x_p, k_p)^\dagger |0\rangle \quad (63)$$

for all $p \geq 1$, and the span of all such states is \mathfrak{H} provided, apart from constant multiples, that $|0\rangle$ is the only state annihilated by all the A operators. Thus

we are led to a conventional Fock representation for the A and A^\dagger operators. Note that the Fock operators are irreducible, and thus all operators acting in \mathfrak{H} are given as suitable functions of them.

Next, let $c(x, k)$ be a possibly complex, c -number function and introduce the translated Fock operators

$$B(x, k) \equiv A(x, k) + c(x, k) \mathbb{1} , \quad (64)$$

$$B(x, k)^\dagger \equiv A(x, k)^\dagger + c(x, k)^* \mathbb{1} . \quad (65)$$

Evidently, the only nonvanishing commutator of the B and B^\dagger operators is

$$[B(x, k), B(x', k')^\dagger] = \delta(x, x') \delta(k, k') \mathbb{1} , \quad (66)$$

the same as the A and A^\dagger operators. With regard to transformations of the coordinate x , it is clear that $c(x, k)$ (just like the local operators A and B) should transform as a scalar density of weight one-half. Thus we set

$$c(x, k) \equiv b(x)^{1/2} d(x, k) , \quad (67)$$

where $d(x, k)$ transforms as a scalar. The criteria for acceptable $d(x, k)$ are, for each x , that

$$\int_+ |d(x, k)|^2 dk = \infty , \quad (68)$$

$$\int_+ k_{rs} |d(x, k)|^2 dk = 2\delta_{rs} , \quad (69)$$

the latter assuming [cf., the discussion following (39)] that $\tilde{g}_{kl}(x) = \delta_{kl}$. In (68) and (69) we have introduced $dk \equiv \Pi_{a \leq b} dk_{ab}$, and the symbol “+” signifies an integration over only those k values for which $\{k_{ab}\} > 0$.

We shall focus on only one particular choice for d , specifically,

$$d(x, k) \equiv \frac{K e^{-\text{tr}(k)}}{\det(k)} , \quad (70)$$

which is everywhere independent of x ; K denotes a positive constant to be fixed later. The given choice for d corresponds to the case where the asymptotic fields $\tilde{\pi}^{kl}(x) \equiv 0$ and $\tilde{g}_{kl}(x) \equiv \delta_{kl}$. **[Remark:** For different choices of asymptotic fields it suffices to choose

$$d(x, k) \rightarrow \tilde{d}(x, k) \equiv \frac{K e^{-ib(x)^{-1} \tilde{\pi}^{ab}(x) k_{ab}} e^{-\tilde{g}^{ab}(x) k_{ab}}}{\det(k)} . \quad (71)$$

We shall not explicitly discuss this case further.]

In terms of these quantities, the local metric operator is defined by

$$\sigma_{ab}(x) \equiv b(x)^{-1} \int_+ B(x, k)^\dagger k_{ab} B(x, k) dk, \quad (72)$$

and the local scale operator is defined by

$$\kappa_s^r(x) \equiv -i \frac{1}{2} \int_+ B(x, k)^\dagger (k_{st} \overrightarrow{\partial}^{tr} - \overleftarrow{\partial}^{rt} k_{ts}) B(x, k) dk. \quad (73)$$

Here $\overrightarrow{\partial}^{st} \equiv \partial/\partial k_{st}$, $\overleftarrow{\partial}^{rt} \equiv \partial/\partial k_{rt}$ acting to the left, and $\sigma_{ab}(x)$ transforms as a tensor while $\kappa_s^r(x)$ transforms as a tensor density of weight one. It is straightforward to show that these operators satisfy the required affine commutation relations, and moreover that [31, 32]

$$\begin{aligned} & \langle 0 | e^{i \int \pi^{ab}(x) \sigma_{ab}(x) d^3x} e^{-i \int \gamma_r^s(x) \kappa_s^r(x) d^3x} | 0 \rangle \\ &= \exp \{ -K^2 \int b(x) d^3x \int [e^{-2\delta^{ab} k_{ab}} - e^{-i\pi^{ab}(x) k_{ab}/b(x)} e^{-[(\delta^{ab} + g^{ab}(x)) k_{ab}]}] dk / (\det k)^2 \} \\ &= \exp [-2 \int b(x) d^3x \ln ([\det(g_{ab}(x))]^{1/2} \det \{ \frac{1}{2} [\delta^{ab} + g^{ab}(x)] - i \frac{1}{2} b(x)^{-1} \pi^{ab}(x) \})] , \end{aligned} \quad (74)$$

where K has been chosen so that

$$K^2 \int_+ k_{rs} e^{-2 \text{tr}(k)} dk / (\det k)^2 = 2 \delta_{rs}. \quad (75)$$

An obvious extension of this calculation leads to (39).

3.6 Local operator products

Basically, local products for the gravitational field operators follow the pattern for other ultralocal quantum field theories [31, 32]. As motivation, consider the product

$$\begin{aligned} & \sigma_{ab}(x) \sigma_{cd}(y) \\ &= b(x)^{-2} \int_+ \int_+ B(x, k)^\dagger k_{ab} [B(x, k), B(y, k')^\dagger] k'_{cd} B(y, k') dk dk' \\ &\quad + : \sigma_{ab}(x) \sigma_{cd}(y) : \\ &= b(x)^{-2} \delta(x, y) \int_+ B(x, k)^\dagger k_{ab} k_{cd} B(x, k) dk + : \sigma_{ab}(x) \sigma_{cd}(y) : , \end{aligned} \quad (76)$$

where $: :$ denotes normal ordering with respect to A and A^\dagger . When $y = x$, this relation formally becomes

$$\sigma_{ab}(x)\sigma_{cd}(x) = b(x)^{-2}\delta(x, x)\int_+ B(x, k)^\dagger k_{ab} k_{cd} B(x, k) dk + : \sigma_{ab}(x)\sigma_{cd}(x) : (77)$$

We define the renormalized (subscript “ R ”) local product

$$[\sigma_{ab}(x)\sigma_{cd}(x)]_R \equiv b(x)^{-1}\int_+ B(x, k)^\dagger k_{ab} k_{cd} B(x, k) dk \quad (78)$$

after formally dividing both sides by the divergent dimensionless “scalar” $b(x)^{-1}\delta(x, x)$.⁵ Higher-order local products exist as well, for example,

$$\begin{aligned} & [\sigma_{a_1 b_1}(x)\sigma^{a_2 b_2}(x)\sigma_{a_3 b_3}(x)\cdots\sigma_{a_p b_p}(x)]_R \\ & \equiv b(x)^{-1}\int_+ B(x, k)^\dagger (k_{a_1 b_1} k^{a_2 b_2} k_{a_3 b_3} \cdots k_{a_p b_p}) B(x, k) dk, \end{aligned} \quad (79)$$

which, after contracting on b_1 and b_2 , implies that

$$[\sigma_{a_1 b}(x)\sigma^{a_2 b}(x)\sigma_{a_3 b_3}(x)\cdots\sigma_{a_p b_p}(x)]_R = \delta_{a_1}^{a_2} [\sigma_{a_3 b_3}(x)\cdots\sigma_{a_p b_p}(x)]_R. \quad (80)$$

It is in this sense that $[\sigma_{ab}(x)\sigma^{bc}(x)]_R = \delta_a^c$.

We take up only one further point regarding local products. It is rather natural [31, 32] to try to define the local momentum “operator” by

$$\mu^{rs}(y) = -i\frac{1}{2}\int_+ B(y, k)^\dagger (\overrightarrow{\partial}^{ab} - \overleftarrow{\partial}^{ab}) B(y, k) dk, \quad (81)$$

but this expression only leads to a form and not a local operator. Furthermore, the putative canonical commutation relation becomes

$$\begin{aligned} [\sigma_{ab}(x), \mu^{rs}(y)] &= i\delta_{ab}^{rs}\delta(x, y)b(x)^{-1}\int_+ B(x, k) B(x, k) dk \\ &= i\delta_{ab}^{rs}\delta(x, y)[\int_+ |d(x, k)|^2 dk + \dots], \end{aligned} \quad (82)$$

which has a divergent multiplier and is, therefore, not even a form. On the other hand, it is true that

$$\begin{aligned} & \frac{1}{2}[\sigma_{rl}(x)\mu^{ls}(x) + \mu^{sl}(x)\sigma_{lr}(x)]_R \\ &= -i\frac{1}{2}\int_+ B(x, k) (k_{rl}\overrightarrow{\partial}^{ls} - \overleftarrow{\partial}^{sl} k_{lr}) B(x, k) dk \\ &= \kappa_r^s(x) \end{aligned} \quad (83)$$

as claimed.

⁵For scalar ultralocal theories, the formal dividing factor is the divergent dimensionless “number” $b^{-1}\delta(0)$, where $b > 0$ is an arbitrary factor with suitable dimensions. For gravity, $b \rightarrow b(x)$, our scalar density of weight one. Note that limits involving test functions offer a rigorous definition of the renormalized product [31, 32].

4 Imposition of Constraints

Gravity has four constraints at every point $x \in \mathcal{S}$, and, when expressed in suitable units, they are the familiar spatial and temporal constraints, all densities of weight one, given by [26]

$$H_a(x) = -g_{ab}(x)\pi^{bc}|_c(x) , \quad (84)$$

$$H(x) = \frac{1}{2}g(x)^{-1/2}[g_{ab}(x)g_{cd}(x) + g_{ad}(x)g_{cb}(x) - 2g_{ac}(x)g_{bd}(x)] \\ \times \pi^{ac}(x)\pi^{bd}(x) + g(x)^{1/2} {}^{(3)}R(x) . \quad (85)$$

The spatial constraints are comparatively easy to incorporate since their generators serve as generators of the diffeomorphism group acting on functions of the canonical variables. Stated otherwise, finite spatial diffeomorphism transformations map any coherent state onto another coherent state as in (41) and (42). However, this is decidedly not the case for the temporal constraint. What follows is an account of what to do about these constraints *in principle*; in Part II on this subject, we will discuss how to accomplish these goals.

One satisfactory procedure to incorporate all the necessary constraints is as follows. Let $\{h_p(x)\}_{p=1}^\infty$ denote a complete, orthonormal set of real functions on \mathcal{S} relative to the weight $b(x)$. In particular, we suppose that

$$\int h_p(x) h_n(x) b(x) d^3x = \delta_{pn} , \quad (86)$$

$$b(x) \sum_{p=1}^\infty h_p(x) h_p(y) = \delta(x, y) . \quad (87)$$

Based on this orthonormal set of functions, we next introduce four infinite sequences of constraints

$$H_{(p)a} \equiv \int h_p(x) H_a(x) d^3x , \quad (88)$$

$$H_{(p)} \equiv \int h_p(x) H(x) d^3x , \quad (89)$$

$1 \leq p < \infty$, all of which vanish in the classical theory.

For the quantum theory let us assume, for each p , that $\mathcal{H}_{(p)a}$ and $\mathcal{H}_{(p)}$ are self adjoint, and even stronger that

$$X_P^2 \equiv \sum_{p=1}^P 2^{-p} [\sum_{a=1}^3 (\mathcal{H}_{(p)a})^2 + (\mathcal{H}_{(p)})^2] \quad (90)$$

is self adjoint for all $P < \infty$. Note well, as one potential example, the factor 2^{-P} introduced as part of a regulator as $P \rightarrow \infty$; we comment on this regulator in the next section. For each $\delta \equiv \delta(\hbar) > 0$, let

$$\mathbb{E}_P \equiv \mathbb{E}(X_P^2 \leq \delta^2) \quad (91)$$

denote a projection operator depending on X_P and δ as indicated. How such projection operators may be constructed is discussed in [7] and will be dealt with in Part II. Let

$$S_P \equiv \limsup_{\pi, g} \langle \pi, g | \mathbb{E}_P | \pi, g \rangle, \quad (92)$$

which satisfies $S_P > 0$ since $\mathbb{E}_P \not\equiv 0$ when restricted to sufficiently large δ . Finally, we define

$$\langle\langle \pi'', g'' | \pi', g' \rangle\rangle \equiv \limsup_{P \rightarrow \infty} S_P^{-1} \langle \pi'', g'' | \mathbb{E}_P | \pi', g' \rangle \quad (93)$$

as a reduction of the original reproducing kernel. The result is either trivial, say if δ is too small, or it leads to a continuous, positive-definite functional on the original phase space variables. We focus on the latter case.

To obtain the final physical Hilbert space, one must study $\langle\langle \pi'', g'' | \pi', g' \rangle\rangle$ as a function of the regularization parameter δ . Since gravity has an anomaly [33], there should be a minimum value of δ , which is still positive, that defines the proper theory, rather like the example in (30). Assuming we can find and then use that value, $\langle\langle \pi'', g'' | \pi', g' \rangle\rangle$ becomes the reproducing kernel for the physical Hilbert space $\mathfrak{H}_{\text{phys}}$. Attaining this goal would then permit the real work of extracting the physics to begin.

Our discussion regarding constraints in this paper has indeed been brief. Although the program we have in mind is not simple, it has the virtue of being realizable, at least in principle. After all, before any calculational scheme is developed it is always wise to ensure that the object under study has a good chance of existing!

5 Discussion

In the preceding sections we have outlined an approach to quantum gravity that it is somewhat different than currently considered. As background to

our philosophy, let us briefly review some of the common weak points in the standard ways of quantizing gravity, and use these comments as motivation for our approach.

5.1 Traditional viewpoints & commentary

(i) Viewed by way of conventional perturbation theory, quantum gravity has two main difficulties of principle. On the one hand, the perturbative split of the metric in the form $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ (or any other background metric), with canonical quantization of the “small” deviation $h_{\mu\nu}(x)$ violates signature properties since in that case the spectrum of $h_{\mu\nu}(x)$ is unbounded above and below. On the other hand, as an asymptotically nonfree theory, gravity is nonrenormalizable and poorly described by a perturbation theory which needs an unending addition of distinct counterterms with divergent coefficients.

To address these obstacles, we first note that the affine approach guarantees a proper metric signature from the very beginning, and second we remind the reader that certain asymptotically nonfree, nonrenormalizable models have indeed been solved [31], and their solution procedures form the core of the present approach to quantize gravity.

(ii) While the constraints of classical gravity are first class, there is an anomaly in the quantum constraints and thus they are effectively second class. Usual views toward second-class constraints involve solving and eliminating them, introducing and then quantizing Dirac brackets, or the conversion of second-class constraints into first-class constraints. Each of these methods is often complicated and not all are guaranteed to be valid beyond a semiclassical treatment if the classical constraint hypersurface has a non-Euclidean geometry [34]. These difficulties have stimulated searches to get around the second-class character altogether, either by introducing non-Hermitian constraint operators that may close algebraically [35], or by introducing additional fields and space-time dimensions until the anomaly cancels.

Regarding these comments, we accept the anomaly and the second-class constraints that it implies. Giving up a classical symmetry is not so heretical as it may seem. For example, Hamiltonian classical mechanics enjoys a full covariance under general canonical coordinate transformations, but that

invariance is *not* preserved *in its classical form* when we go to the quantum theory. For example, consider the classical Poisson brackets for a set of generator elements

$$\{e^{ap+bq}, e^{cp+dq}\} = (bc - ad) e^{(a+c)p+(b+d)q}, \quad (94)$$

where a, b, c , and d are parameters, while in canonical quantum mechanics we have the corresponding commutator algebra

$$[e^{aP+bQ}, e^{cP+dQ}] = (2i) \sin[(bc - ad)/2] e^{(a+c)P+(b+d)Q}. \quad (95)$$

These expressions agree in their algebraic structure for selected elements, but not for the whole algebra. Equivalence begins to break down at the quadratic level, which is exactly the case for the temporal constraint in gravity. In particle mechanics there are sound physical reasons [36] for this “breakdown” of symmetry, and attempts to restore the symmetry—as in geometric quantization [37]—go counter to such sound physical principles. There is no reason that a similar scenario does not hold for gravity. The breakdown of the classical symmetry and the appearance of a quantum “anomaly” (better called a “quantum mechanical symmetry breaking” [38]) could, just as in the quantum mechanics case, carry real physics.

Accepting the second-class nature of (part of) the constraints of quantum gravity means a different approach must be taken. As already noted, earlier approaches required solving for and eliminating the unphysical variables, the introduction of Dirac brackets, etc., all of which are rather technical and may be extremely complicated. In the present view, afforded by the projection operator approach to constraints—*second-class constraints included*—none of these particular complications arise. Instead, one projects onto the state (or, with degeneracy, states) for which the sum of the square of the constraints is bounded. Why the square and not the fourth power? Using the fourth power would not be wrong; the only change would involve a unitary transformation of the original result, which maps one set of “ground” states onto another set of “ground” states. The square is chosen for simplicity, not for any reasons of exclusivity.

5.2 Relation to earlier work

Pilati, in a series of papers [3] (see also [2]), analyzed a strong coupling model of quantum gravity in which the temporal constraint given in (85)

was modified to read

$$H'(x) = \frac{1}{2}g(x)^{-1/2}[g_{ab}(x)g_{cd}(x) + g_{ad}(x)g_{cb}(x) - 2g_{ac}(x)g_{bd}(x)]\pi^{ac}(x)\pi^{bd}(x) , \quad (96)$$

namely, the second term involving the scalar curvature ${}^{(3)}R(x)$ based on the metric $g_{ab}(x)$ was dropped. The reason for doing so was to achieve a theory in which the temporal constraint $H'(x)$ itself was patterned after the Hamiltonian density of an ultralocal theory. This modification was thought to be advantageous because then all the machinery developed for ultralocal quantum field theory could be used for the strong coupling gravitational model. Once that model was under control, it was the hope to reintroduce the dropped term by a perturbation theory analysis. Unfortunately, the reintroduction of dropped terms involving spatial gradients has never been successfully accomplished by a perturbation analysis about a non-Gaussian ultralocal model. This failure is most likely because such “interaction terms” generally amount to nonrenormalizable perturbations of the unperturbed (ultralocal) models.

The program advanced in the present paper takes a different view toward these issues.

First, we focus on kinematics with the knowledge that for pure gravity the Hamiltonian operator vanishes, as it does in any situation that is reparametrization invariant. In its place we find constraints, and the real physical content of the theory lies in the particular constraints. However, before the constraints can be introduced, there must be a “primary container” to receive them. In our case, this primary container is the Hilbert space and set of relevant operators prior to the introduction of *any* form of the constraints, and which is based on the fundamental physical nature of the variables, i.e., positive-definite, 3×3 matrix-valued, local field operators, etc. This is the preferred procedure: *Quantization before the introduction of any constraints*. At this primary level, there is no coupling of one degree of freedom with another—any coupling comes through the enforcement of specific constraints. Hence, in the primary container the degrees of freedom are mutually independent of each other. For finitely many kinematical degrees of freedom this means that the Hilbert space is a product over spaces for each of the separate degrees of freedom; in a field theory, this independence means that the kinematical operators enter as ultralocal field operators. Consequently, even though the several constraint operators waiting to be in-

roduced may themselves *not* be ultralocal in nature, the primary container itself, which has been prepared to receive them, is ultralocal.

At this point the reader may wish to reexamine (39)—in essence our “primary container”—to recall the appearance of an ultralocal state on a set of field operators. Apart from the 3×3 matrix character, the functional form of (39) emerges from (i) the product of N expressions of the form (16) for independent arguments p_n, q_n, r_n, s_n , $1 \leq n \leq N$, (ii) the replacement of β by $b_n \Delta$ and $(p_n - r_n)$ by $(p_n - r_n) \Delta$, and (iii) the limit as $N \rightarrow \infty$, $\Delta \rightarrow 0$ such that $\Sigma(\cdot) b_n \Delta \rightarrow \int(\cdot) b(x) dx$. In this way we have created, from a collection of independent single affine degrees of freedom, the reproducing kernel for affine gravity in $1 + 1$ dimensional space. In a similar manner, a set of independent 3×3 affine degrees of freedom can be (and were) used to build an ultralocal representation for 3×3 metric and momentum fields in (39); moreover, this type of construction does not favor the “natural” phase-space metric (61). In summary, we emphasize that whenever the “dynamics” appears through constraints, the primary container should be ultralocal in character. We next turn our attention to the introduction of the constraints.

In the projection operator approach it is recognized from the outset that the physical Hilbert space—or better the *regularized* physical Hilbert space—is a *subspace* of the original Hilbert space that is uniquely determined by an associated projection operator \mathbb{E} . Whatever form the constraints may take, they are “encoded” into the projection operator \mathbb{E} , and a regularization means that the constraints are satisfied to a certain level of precision determined by a regularization parameter δ . How to turn constraint operators into projection operators in general has been discussed in [7]; as regards the gravitational case, that project will be discussed in Part II.

Continuing still in a general framework, let us consider an expression that may be used either to generate dynamics or to enforce constraints. From a classical point of view, and especially from a path integral point of view, it may seem that quantities used in either of these ways may be rather similar. However, it is important to already understand that there is a fundamental distinction between the use of a quantum operator either (i) to generate unitary transformations or (ii) to serve as a constraint operator in a given system. In the first case, the operator must be self adjoint and thus densely defined, while in the second case the operator may be defined on only the zero vector! This fact has profound consequences. In particular, to have a self-adjoint generator requires that the operator representation in the primary

container must already be finely “tuned” to ensure that the generator which will be introduced is self adjoint (as in Haag’s Theorem [39]). For constraint operator imposition this need not be the case, and the reason this is so is because we allow for changes, i.e., adaptations, of the primary container representations through the process of reduction of the reproducing kernel. As an example, let us consider only the local temporal operator $\mathcal{H}(x)$ for gravity. On the one hand, to generate unitary time evolutions it may be necessary that $\int \mathcal{H}(x) d^3x$ be self adjoint. On the other hand, to enforce constraints, it is only necessary that $\mathcal{H}_{(p)}$ [cf., (89)] be “small”, but there is no requirement that these operators must be *uniformly* “small”. Instead they can be “small” in the sense that $X^2 \equiv \sum_{p=1}^{\infty} s_p (\mathcal{H}_{(p)})^2$ is “small”, where the set of positive constants $\{s_p\}$ serve as regulators to control convergence of the series. The example chosen was that $s_p = 2^{-p}$, but there is nothing special about that choice. Any reasonable choice that leads to a self adjoint operator X should lead to the same reproducing kernel in the final analysis when the regularization parameter δ attains its final value for the problem at hand.

A rather simple example of the general procedure discussed above can be seen in studies involving product representations [40]. Additionally, it is instructive to reanalyze the relativistic free field by this procedure to see how ultralocal representations turn into nonultralocal representations.

Suffice it to say, it is this vast difference between the required nature of constraint operators and unitary generators that permits us to start with ultralocal field operator representations and emerge with *nonultralocal* operators in the physical Hilbert space.

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